IAS 2019 SUMMER COLLABORATORS REPORT: ALGEBRAIC COBORDISM, HILBERT SCHEMES, AND DERIVED ALGEBRAIC GEOMETRY

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The goal of this collaboration is to make progress towards an algebro-geometric refinement of the deep connection between stable homotopy theory and the geometry of smooth manifolds. In its basic form, this connection was discovered by Pontryagin, Thom, and Quillen, who gave manifold-theoretic descriptions of the stable homotopy groups and cohomology groups of Thom spectra. Their computations were substantially refined in the work of Galatius, Madsen, Tillmann, and Weiss [GMTW09], who identified the infinite loop spaces of Thom spectra with cobordism spaces of structured manifolds. Both sides of this identification have natural counterparts in algebraic geometry:

- The algebro-geometric analog of a Thom spectrum is known as a *motivic Thom spectrum*; a central example is Voevodsky's motivic Thom spectrum MGL, which is a refinement of MU.
- The algebro-geometric analog of the cobordism space of a class of manifolds is the $\mathbb{A}^1$-*homotopy type* of a moduli stack of schemes.

In this report we focus our attention on the motivic spectrum MGL, although Theorem 1 below extends to more general motivic Thom spectra (such as the motivic sphere spectrum).

During the two weeks we spent at IAS in July 2019 we finalized the following theorem, which has since appeared in the preprint [EHK$^+$19] and is the basis of our investigation:

Theorem 1. Let k be a field. Then $\Omega^\infty_T\mathrm{MGL}$ is the $\mathbb{A}^1$-homotopy type of the group completion of the moduli stack of finite syntomic k-schemes. If $n > 0$, then $\Omega^\infty_T\Sigma^n_T\mathrm{MGL}$ is the $\mathbb{A}^1$-homotopy type of the moduli stack of finite quasi-smooth derived k-schemes of virtual dimension $-n$.

We also proved the following crucial result, which replaces the group completion in the above description of $\Omega^\infty_T\mathrm{MGL}$ by Quillen's plus construction:

Theorem 2. Over any field k, there is a motivic equivalence $\Omega^\infty_T\mathrm{MGL} \simeq \mathbb{Z} \times \mathrm{Hilb}^{\mathrm{lci}}_\infty(\mathbb{A}^\infty)^+$, where $+$ is Quillen's plus construction.

Here, $\mathrm{Hilb}^{\mathrm{lci}}_\infty(\mathbb{A}^\infty) = \mathrm{colim}_{d,n\to\infty}\mathrm{Hilb}^{\mathrm{lci}}_d(\mathbb{A}^n)$ and $\mathrm{Hilb}^{\mathrm{lci}}_d(\mathbb{A}^n) \subset \mathrm{Hilb}_d(\mathbb{A}^n)$ is the open subscheme of local complete intersections in the Hilbert scheme of d points in $\mathbb{A}^n$. The content of this theorem is a moving lemma that says that every finite syntomic k-scheme is stably (with respect to the degree) cobordant to the split cover of the same degree. This result is similar to Morel and Voevodsky's computation $\Omega^\infty_T\mathrm{KGL} \simeq \mathbb{Z}\times\mathrm{Gr}_\infty(\mathbb{A}^\infty)$, except that we cannot remove the plus construction. However, since the plus construction is invisible to motives, Theorem 2 implies

$$M(\Omega^\infty_T\mathrm{MGL}) \simeq \bigoplus_{d\in\mathbb{Z}} M(\mathrm{Hilb}^{\mathrm{lci}}_\infty(\mathbb{A}^\infty)).$$

The "Wilson space hypothesis" formulated by Mike Hopkins predicts that the left-hand side is a pure Tate motive, i.e., a sum of motives of the form $\mathbb{Z}(i)[2i]$. Motivated by this conjecture, we started the computation of the motives of lci Hilbert schemes in low degrees. The basic idea is to use a linear action of a 1-dimensional torus $\mathbb{G}_m$ on $\mathbb{A}^n$ to obtain a cellular decomposition of $\mathrm{Hilb}^{\mathrm{lci}}_d(\mathbb{A}^n)$, using a method introduced by Białynicki-Birula. In degree 2, where all points are lci, we can easily compute

$$M(\mathrm{Hilb}^{\mathrm{lci}}_2(\mathbb{A}^n)) \simeq \bigoplus_{0\le i\le n-1} \mathbb{Z}(i)[2i].$$

The Białynicki-Birula method does not directly apply to $\mathrm{Hilb}^{\mathrm{lci}}_d(\mathbb{A}^n)$ for $d \geq 3$, as there are examples of lci points converging to non-lci points under the torus action. Our strategy was therefore to analyze the Białynicki-Birula stratification of the larger Hilbert scheme $\mathrm{Hilb}_d(\mathbb{A}^n)$ and try to identify the induced stratification of the open lci locus $\mathrm{Hilb}^{\mathrm{lci}}_d(\mathbb{A}^n)$. Together with Joachim Jelisiejew, we did this successfully for $d = 3$:

Theorem 3.

$$M(\mathrm{Hilb}_3^{\mathrm{lci}}(\mathbb{A}^n)) \simeq \bigoplus_{0\leq i\leq n-1} \mathbb{Z}(2i)[4i] \oplus \bigoplus_{0\leq i<j\leq n-1} \left(\mathbb{Z}(i+j)[2i+2j] \oplus \mathbb{Z}(i+j+3)[2i+2j+5]\right).$$

The indices in the second sum are in natural correspondence with the Schubert cells in the Grassmannian $\mathrm{Gr}(2,n)$. This computation already shows that $M(\mathrm{Hilb}_{\leq 3}^{\mathrm{lci}}(\mathbb{A}^\infty))$ is mixed Tate but not pure Tate, thus ruling out a naive generalization of the Wilson space hypothesis. Nevertheless, this gives some evidence that $\mathrm{Hilb}_\infty^{\mathrm{lci}}(\mathbb{A}^\infty)$ has a mixed Tate motive. In characteristic zero, this should suffice to deduce that it has a pure Tate motive, thanks to Hill and Hopkins' proof of the analog of the Wilson space hypothesis in C_2-equivariant homotopy theory [HH18]. We also attempted to reduce these computations to the punctual Hilbert schemes, but this turns out to require understanding the motive of the configuration space of d points in $\mathbb{A}^n$, which seems at least as difficult as the original problem.

In a different direction, we hope to eventually generalize Theorem 1 to include the case $n<0$. The result cannot hold as stated since a finite derived scheme has virtual dimension at most 0, but we expect a version of the result with "finite" replaced by "projective". This is however already a nontrivial modification for $n\geq 0$. Let $\mathcal{FQS}\mathrm{m}_k^n$ (resp. $\mathcal{PQS}\mathrm{m}_k^n$) denote the moduli stack of finite (resp. projective) quasi-smooth k-scheme of virtual dimension $-n$. Then we have an inclusion $\mathcal{FQS}\mathrm{m}_k^n \subset \mathcal{PQS}\mathrm{m}_k^n$. Both stacks are commutative monoids under disjoint union. A first interesting result is the following:

Theorem 4. For every $n\in\mathbb{Z}$, the $\mathbb{A}^1$-homotopy type of the commutative monoid $\mathcal{PQS}\mathrm{m}_k^n$ is a *group*.

This restores the full analogy with the topological situation, where cobordism spaces are always groups under disjoint union. To prove this theorem, we consider 0-dimensional derived structures on the projective line $\mathbb{P}^1$: for every $r\in\mathbb{Z}$, let $\mathbb{P}^1(r)$ be the zero locus of the zero section of the line bundle $\mathcal{O}(r)$ on $\mathbb{P}^1$. Then we construct algebraic cobordisms $\mathbb{P}^1(r)\sim r$ for $r\geq 0$ and $\mathbb{P}^1(r)\sqcup\mathbb{P}^1(s)\sim\mathbb{P}^1(r+s)$ for all $r,s\in\mathbb{Z}$. In particular, $\mathbb{P}^1(-1)$ is an additive inverse of 1 in the graded semiring $\coprod_{n\in\mathbb{Z}} L_{\mathbb{A}^1}\mathcal{PQS}\mathrm{m}_k^n$, which is therefore a ring.

Further investigation led us to formulate the following conjecture, which we are able to prove modulo a direct summand:

Conjecture 5. For every $n\geq 0$, the $\mathbb{A}^1$-homotopy type of $\mathcal{PQS}\mathrm{m}_k^n$ is the group completion of that of $\mathcal{FQS}\mathrm{m}_k^n$.

The main obstruction to proving this conjecture is that we do not understand well the global structure of projective quasi-smooth derived schemes. It might be that it is necessary to impose further conditions on the objects in $\mathcal{PQS}\mathrm{m}_k^n$, such as being cut out by homogeneous equations in a projective space (which is automatic for classical projective schemes but not for derived projective schemes). With this restriction, we expect that the conjecture can be proved by a generic translation argument.

References

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