REMARKS ON ÉTALE MOTIVIC STABLE HOMOTOPY THEORY

TOM BACHMANN, WITH AN APPENDIX BY TOM BACHMANN AND MARC HOYOIS

Contents

1. Introduction

We strengthen some results in étale (and real étale) motivic stable homotopy theory, by eliminating finiteness hypotheses, additional localizations and/or extending to spectra from $H\mathbb{Z}$-modules. The main results are Theorem 3.1 (proving rigidity in ℓ-adic étale motivic stable homotopy theory for any $\mathbb{Z}[1/\ell]$-scheme), Theorem 4.2 (proving rigidity in real étale motivic stable homotopy theory for any scheme) and Theorem A.1 (proving p-periodicity in étale motivic stable homotopy theory of characteristic p-schemes).

Notation and conventions. Given a scheme S, we denote by $S_{ét}^{\wedge}$ its hypercompleted étale ∞-topos and by $\mathrm{Sp}(S_{ét}^{\wedge})$ the stabilization thereof. In contrast, we denote by $\mathcal{SH}_{ét}^{\wedge}(S)$ the localization of the motivic stable category $\mathcal{SH}(S)$ at the (desuspended) étale hypercovers, and similarly denote by $\mathcal{SH}_{ét}^{\wedge S^1}(S)$ the localization of motivic S^1-spectra. We also write $\mathrm{Sp}(\mathcal{S}\mathrm{m}_{S,ét})$ for the category of spectral étale sheaves on the site $\mathcal{S}\mathrm{m}_S$, and $\mathrm{Sp}(\mathcal{S}\mathrm{m}_{S,ét}^{\wedge})$ for its hypercompletion. Given a stable ∞-category $\mathcal{C}$, we denote by $\mathcal{C}_p^{\wedge}$ the subcategory of p-complete objects.

We freely use the language and notation of [Lur17b, Lur17a].

2. Invertible retracts of symmetric objects

Proposition 2.1. *Let $\mathcal{C}$ be a symmetric monoidal ∞-category, $I \in \mathcal{C}$ invertible and $I \xrightarrow{e} X \xrightarrow{f} I$ a retraction. If X is symmetric (i.e. the cyclic permutation on $X^{\otimes n}$ is homotopic to the identity for some $n \geq 2$) then e and f are inverse equivalences.*

Proof. Since a $(2n-1)$-cycle is a product of two n-cycles, we may assume that X is n-symmetric where n is odd; then also I^{-1} is n-symmetric [Dug14, Lemma 4.17]. It follows that $X \otimes I^{-1}$ is n-symmetric, whence (replacing X by $X \otimes I^{-1}$) we may assume that $I = \mathbb{1}$.

First we offer a slightly simpler proof in the case that $\mathcal{C}$ is semiadditive and idempotent complete, and the tensor product distributes over sums. In this case we can write $X \simeq \mathbb{1} \oplus X'$, and so $X^{\otimes n} \simeq \mathbb{1} \oplus X'^{\oplus n} \oplus \dots$, where the symmetric group action restricts to the summand $X'^{\oplus n}$ and yields the canonical action for the $\oplus$ symmetric monoidal structure. In particular X' is $\oplus$-symmetric, which implies that $1 = 0$ as endomorphisms of X', i.e. $X' = 0$ as desired.

In general, consider the maps

$$e' : X \simeq X \otimes \mathbb{1}^{\otimes n-1} \xrightarrow{\mathrm{id} \otimes e^{\otimes n-1}} X^{\otimes n} \quad \text{and} \quad f' : X^{\otimes n} \xrightarrow{\mathrm{id} \otimes f^{\otimes n-1}} X \otimes \mathbb{1}^{\otimes n-1} \simeq X.$$

Then $f'e' \simeq \mathrm{id} \otimes \mathrm{id}^{\otimes n-1} \simeq \mathrm{id}$. On the other hand if σ is the cyclic permutation of $X^{\otimes n}$, then $f'\sigma e'$ is the tensor product of $f : X \to \mathbb{1}$, $n-2$ copies of $\mathrm{id} : \mathbb{1} \to \mathbb{1}$ and $e : \mathbb{1} \to X$, which (up to unit equivalences) is the same as ef. Consequently if $\sigma \simeq \mathrm{id}$ then $\mathrm{id} \simeq f'e' \simeq f'\sigma e' \simeq ef$, so that e, f are indeed inverse equivalences. □

Example 2.2. Since invertible objects are symmetric [Dug14, Lemma 4.17], Proposition 2.1 strengthens [Bac18, Lemma 30].

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Example 2.3. Since $S^{2,1} \in \mathcal{S}\mathrm{pc}(S)_*$ is 3-symmetric (see e.g. [Hoy17, Lemma 6.3]), $\mathbb{G}_m \simeq S^{-1} \wedge S^{2,1} \in \mathcal{SH}^{S^1}(S)$ is also 3-symmetric.

3. Étale topology

The following result was proved in [Bac20, Theorem 6.5] under additional finiteness assumptions, and with an additional localization on the middle category.

Theorem 3.1. *Let S be a scheme with $\ell \in \mathcal{O}_S^\times$. Then the canonical functors*

$$\mathrm{Sp}(S_{\acute{e}t}^\wedge)_\ell^\wedge \to \mathcal{SH}_{\acute{e}t}^{\wedge S^1}(S)_\ell^\wedge \to \mathcal{SH}_{\acute{e}t}^\wedge(S)_\ell^\wedge$$

are equivalences.

We can also remove finiteness assumptions from related rigidity results. For example, denote by $D(S_{\acute{e}t}^\wedge, \mathbb{Z}/\ell)$ the unbounded derived category sheaves of $\mathbb{Z}/\ell$-modules, and by $DA_{\acute{e}t}^\wedge(S, \mathbb{Z}/\ell)$ Ayoub's category of étale motives without transfers [Ayo14, §3]. The equivalence of the outer two categories in the following result was proved in [Ayo14, Theoreme 4.1] under additional finiteness assumptions.

Corollary 3.2. *Let S be a scheme with $\ell \in \mathcal{O}_S^\times$. Then the canonical functors*

$$D(S_{\acute{e}t}^\wedge, \mathbb{Z}/\ell) \to DA_{\acute{e}t}^{\wedge S^1}(S, \mathbb{Z}/\ell) \to DA_{\acute{e}t}^\wedge(S, \mathbb{Z}/\ell)$$

are equivalences.

Proof. Writing $H\mathbb{Z}/\ell \in \mathrm{CAlg}(\mathrm{Sp}(S_{\acute{e}t}^\wedge)_\ell^\wedge)$ for the Eilenberg–MacLane spectrum, the result follows from the equivalences $D(S_{\acute{e}t}^\wedge, \mathbb{Z}/\ell) \simeq \mathrm{Mod}_{H\mathbb{Z}/\ell}(\mathrm{Sp}(S_{\acute{e}t}^\wedge)_\ell^\wedge)$, $DA_{\acute{e}t}^{\wedge S^1}(S, \mathbb{Z}/\ell) \simeq \mathrm{Mod}_{H\mathbb{Z}/\ell}(\mathcal{SH}_{\acute{e}t}^{\wedge S^1}(S)_\ell^\wedge)$ and $DA_{\acute{e}t}^\wedge(S, \mathbb{Z}/\ell) \simeq \mathrm{Mod}_{H\mathbb{Z}/\ell}(\mathcal{SH}_{\acute{e}t}^\wedge(S)_\ell^\wedge)$. The third is a formal consequence of the second, and the first two follow from [Lur18, Theorem 2.1.2.2]. □

We recall some ingredients used in the proof of Theorem 3.1.

(0) If $F : \mathcal{C} \to \mathcal{D}$ is a cocontinuous, symmetric monoidal functor of presentably symmetric monoidal, stable ∞-categories and $\mathcal{C}$ is ℓ-complete (i.e. $\mathcal{C} \simeq \mathcal{C}_\ell^\wedge$), then so is $\mathcal{D}$. Indeed for objects $X, Y \in \mathcal{D}$, the functor $F(-) \otimes X : \mathcal{C} \to \mathcal{D}$ admits a right adjoint r_X, and hence $\mathrm{map}_{\mathcal{D}}(X, Y) \simeq \mathrm{map}_{\mathcal{C}}(\mathbb{1}, r_X Y)$ is ℓ-complete as needed. In particular, ℓ-completion commutes with localization of stable, presentably symmetric monoidal ∞-categories.

(1) The functors

$$\mathrm{Sp}(S_{\acute{e}t}^\wedge)_\ell^\wedge, \mathcal{SH}_{\acute{e}t}^{\wedge S^1}(S)_\ell^\wedge, \mathcal{SH}_{\acute{e}t}^\wedge(S)_\ell^\wedge : \mathcal{S}\mathrm{ch}_{\mathbb{Z}[1/\ell]}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty$$

are Zariski sheaves. This implies that they are right Kan extended from their restriction to affine $\mathbb{Z}[1/\ell]$-schemes (see e.g. [Hoy15, Lemma C.3]), which is what we shall use. The descent properties are established by arguments entirely analogous to e.g. [Hoy17, Proposition 4.8] [AGV20, §2.3]; in fact all three functors satisfy étale hyperdescent.

(2) The functor

$$\mathcal{E}\mathrm{t}_{(-)}^{\mathrm{fp}} : \mathcal{S}\mathrm{ch}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty$$

(sending X to the category of finitely presented étale X-schemes) is *continuous*: it converts cofiltered limits of quasi-compact quasi-separated schemes with affine transition maps into colimits [GAV72, Lemme VII.5.6]. This implies that also

$$\mathcal{P}(\mathcal{E}\mathrm{t}_{(-)}^{\mathrm{fp}}) : \mathcal{S}\mathrm{ch}^{\mathrm{op}} \to Pr^L$$

is continuous. From this one deduces the same result for spectral presheaves. The category of étale *sheaves* is obtained by inverting the nerves of (finitely presented) étale covers, which must be pulled back from a finite stage by continuity of $\mathcal{E}\mathrm{t}_{(-)}$ and quasi-compactness. It follows that

$$\mathrm{Sp}((-)_{\acute{e}t}) : \mathcal{S}\mathrm{ch}^{\mathrm{op}} \to Pr^L$$

is continuous. Beware the absence of hypercompletion! By similar arguments, the functor

$$\mathcal{SH}_{\acute{e}t}^{S^1}(-) : \mathcal{S}\mathrm{ch}^{\mathrm{op}} \to Pr^L$$

is continuous (again no hypercompletion). From this one easily deduces (e.g. using (0)) that also $\mathrm{Sp}((-)_{\acute{e}t})_\ell^\wedge, \mathcal{SH}_{\acute{e}t}^{S^1}(-)_\ell^\wedge$ are continuous.

(3) Finally recall the object $\hat{\mathbb{1}}_\ell(1)[1] \in \mathrm{Sp}(S_{\acute{e}t}^\wedge)_\ell^\wedge$ and the map $\sigma : \mathbb{G}_m \to \hat{\mathbb{1}}_\ell(1)[1] \in \mathcal{SH}_{\acute{e}t}^{\wedge S^1}(S)_\ell^\wedge$ from [Bac20, §3].

Proof of Theorem 3.1. We first show that $\sigma : \mathbb{G}_m \to \hat{\mathbb{1}}_\ell(1)[1] \in \mathcal{SH}_{\acute{e}t}^{\wedge S^1}(S)^\wedge_\ell$ is an equivalence. By stability of σ under base change, we reduce to $S = Spec(\mathbb{Z}[1/\ell])$, and by [Bac20, Corollary 5.12] we reduce to $S = Spec(k)$, where k is a separably closed field. In this situation $\sigma : \mathbb{G}_m \to \hat{\mathbb{1}}_\ell(1)[1]$ admits a section [Bac20, proof of Theorem 6.5], and thus σ is an equivalence by Proposition 2.1 and Example 2.3.

We have thus proved that $\mathcal{SH}_{\acute{e}t}^{\wedge S^1}(S)^\wedge_\ell \simeq \mathcal{SH}^\wedge_{\acute{e}t}(S)^\wedge_\ell$. In particular the theorem holds whenever [Bac20, Theorem 6.6] applies, so for example $(*)$ if S is finite type over $\mathbb{Z}[1/\ell]$ or the spectrum of a separably closed field of characteristc $\neq \ell$.

To prove that $\mathrm{Sp}(S^\wedge_{\acute{e}t})^\wedge_\ell \simeq \mathcal{SH}_{\acute{e}t}^{\wedge S^1}(S)^\wedge_\ell$, we may assume by right Kan extension that S is affine. In this case we can write $S = \lim_\alpha S_\alpha$, where each S_α is affine (so in particular quasi-compact quasi-separated) and of finite type over $\mathbb{Z}[1/\ell]$. Denote by V_α (respectively V) the class of ∞-connective maps in $\mathrm{Sp}(S_{\alpha,\acute{e}t})$ (respectively $\mathrm{Sp}(S_{\acute{e}t})$) and by W_α (respectively W) the ∞-connective maps in $\mathrm{Sp}(\mathcal{S}\mathrm{m}_{S_\alpha,\acute{e}t})$ (respectively $\mathrm{Sp}(\mathcal{S}\mathrm{m}_{S,\acute{e}t})$). Then $\mathrm{Sp}(S^\wedge_{\alpha,\acute{e}t})^\wedge_\ell \simeq \mathrm{Sp}(S_{\alpha,\acute{e}t})^\wedge_\ell[V_\alpha^{-1}]$ (in other words, $\mathrm{Sp}(S^\wedge_{\alpha,\acute{e}t})^\wedge_\ell$ is the initial object of $Pr^L_{\mathrm{Sp}(S^\wedge_{\alpha,\acute{e}t})^\wedge_\ell/}$ inverting V_α), and similarly $\mathcal{SH}_{\acute{e}t}^{\wedge S^1}(S_\alpha)^\wedge_\ell \simeq \mathcal{SH}_{\acute{e}t}^{S^1}(S_\alpha)^\wedge_\ell[W_\alpha^{-1}]$. By continuity, $\mathrm{Sp}(S_{\acute{e}t})^\wedge_\ell \simeq \mathrm{colim}_\alpha \mathrm{Sp}(S_{\alpha,\acute{e}t})^\wedge_\ell$. Consider the commutative diagram in Pr^L

$$\begin{array}{ccccc}
\mathrm{Sp}(S_{\acute{e}t})^\wedge_\ell & \xrightarrow{a} & \mathrm{colim}_\alpha \mathrm{Sp}(S^\wedge_{\acute{e}t})^\wedge_\ell[V_\alpha^{-1}] & \xrightarrow{b} & \mathrm{Sp}(S^\wedge_{\acute{e}t})^\wedge_\ell \\
\downarrow & & c\downarrow & & d\downarrow \\
\mathcal{SH}_{\acute{e}t}^{S^1}(S)^\wedge_\ell & \xrightarrow{a'} & \mathrm{colim}_\alpha \mathcal{SH}_{\acute{e}t}^{S^1}(S)^\wedge_\ell[W_\alpha^{-1}] & \xrightarrow{b'} & \mathcal{SH}_{\acute{e}t}^{\wedge S^1}(S)^\wedge_\ell.
\end{array}$$

The morphism a is a localization (namely at the union of the pullbacks to S of the classes V_α), and ba is a localization (namely at V); hence so is b. Similarly b' is a localization. The morphism c is an equivalence, being a colimit of equivalences by $(*)$. We deduce that d is a localization. To conclude the proof, it thus suffices to prove that d is conservative. Since equivalences of hypercomplete sheaves may be tested on stalks (essentially by definition of hypercompleteness, this reduces to the well-known special case of sheaves of abelian groups), an object of $\mathrm{Sp}(S^\wedge_{\acute{e}t})^\wedge_\ell$ vanishes if and only if its image under the pullback to $\mathrm{Sp}(\bar{s}^\wedge_{\acute{e}t})^\wedge_\ell$ vanishes, for every geometric point $\bar{s} = Spec(\bar{k}) \to S$ (where $\bar{k}$ is a separably closed field). Since formation of d is natural in S, we are reduced to the case $S = \bar{s}$, which was already established (see $(*)$). □

Corollary 3.3. *Let S be a scheme with $\ell \in \mathcal{O}_S^\times$. Then $\mathrm{Sp}(S^\wedge_{\acute{e}t})^\wedge_\ell \to \mathrm{Sp}((S \times \mathbb{A}^1)^\wedge_{\acute{e}t})^\wedge_\ell$ is fully faithful. (In other words, "ℓ-adic hyper-étale cohomology with spectral coefficients is $\mathbb{A}^1$-invariant".)*

Proof. This holds for $\mathcal{SH}^\wedge_{\acute{e}t}(S)^\wedge_\ell$ essentially by construction. □

4. Real étale topology

Recall the real étale topology from [Sch94]. Denote by $S_{r\acute{e}t}$ the small real étale ∞-topos of S (not hypercompleted), by $\mathcal{SH}_{r\acute{e}t}(S)$ the localization of $\mathcal{SH}(S)$ at the real étale covers, and so on.

Remark 4.1. If $\dim S < \infty$, then $S_{r\acute{e}t}$ and $\mathcal{S}\mathrm{m}_{S,r\acute{e}t}$ are hypercomplete [ES19, Theorem B.13]. It follows that in this situation, our notation coincides with the one from [Bac18] (where everything is hypercompleted and finite dimensional by definition).

The following result strengthens [Bac18, Theorem 35], by removing ρ-inversion from $\mathcal{SH}^{S^1}_{r\acute{e}t}(S)$ and finiteness assumptions from S.

Theorem 4.2. *Let S be any scheme. Then*

$$\mathcal{SH}(S)[\rho^{-1}] \simeq \mathcal{SH}_{r\acute{e}t}(S) \simeq \mathcal{SH}^{S^1}_{r\acute{e}t}(S) \simeq \mathrm{Sp}(S_{r\acute{e}t}).$$

In particular $\rho : S^1 \to \mathbb{G}_m \in \mathcal{SH}^{S^1}_{r\acute{e}t}(S)$ is an equivalence.

Proof. By Zariski descent and continuity (see §3 and [Sch94, proof of Proposition 3.4.1]), we may assume that S is finite type over $\mathbb{Z}$. In this case by [Bac18, Theorem 35], only the last statement requires proof. The proof of [Bac18, Proposition 29] constructs a retraction $S^0 \xrightarrow{\rho} \mathbb{G}_m \to S^0 \in \mathcal{S}\mathrm{pc}_{ret}(S)_*$. The result thus follows from Proposition 2.1 and Example 2.3. □

Appendix A. Vanishing of $\mathcal{SH}^\wedge_{\acute{e}t}(\mathbb{F}_p)^\wedge_p$

Theorem A.1. *We have $\mathbb{1}/p \simeq 0 \in \mathcal{SH}_{\acute{e}t}^{\wedge S^1}(\mathbb{F}_p)$. In particular if $X \in \mathcal{S}\mathrm{ch}_{\mathbb{F}_p}$ then*

$$\mathcal{SH}_{\acute{e}t}^{\wedge S^1}(X)^\wedge_p = * = \mathcal{SH}^\wedge_{\acute{e}t}(X)^\wedge_p.$$

Before the proof, we need some preparation. The category $\mathrm{Sp}(\mathcal{S}\mathrm{m}^{\wedge}_{S,ét})$ of étale hypersheaves of spectra on $\mathcal{S}\mathrm{m}_S$ admits a canonical non-degenerate t-structure (see e.g. [Bac20, §2.2]). Denote by $L^{\wedge}_{ét,\mathrm{mot}}$ the localization endofunctor of $\mathrm{Sp}(\mathcal{S}\mathrm{m}^{\wedge}_{S,ét})$ corresponding to the $\mathbb{A}^1$-equivalences, so that the category of local objects is $\mathcal{SH}^{\wedge S^1}_{ét}(S)$.

Lemma A.2. *If $E \in \mathrm{Sp}(\mathcal{S}\mathrm{m}^{\wedge}_{\mathbb{F}_p,ét})_{\geq 0}$, then $L^{\wedge}_{ét,\mathrm{mot}}E/p \in \mathrm{Sp}(\mathcal{S}\mathrm{m}^{\wedge}_{\mathbb{F}_p,ét})_{\geq -1}$.*

Proof. Denote by $L_{\mathbb{A}^1}E$ the presheaf

$$X \mapsto \operatorname*{colim}_{n\in\Delta^{\mathrm{op}}} E(X \times \mathbb{A}^n).$$

Then $L_{\mathbb{A}^1}E$ is $\mathbb{A}^1$-invariant and $E \to L_{\mathbb{A}^1}E$ is an $\mathbb{A}^1$-equivalence [MV99, Corollaries 2.3.5 and 2.3.8]. Moreover since $\mathrm{cd}(\mathbb{F}_p) < \infty$, étale hypersheaves are closed under colimits in presheaves (see e.g. [Bac20, Lemma 2.16]), and thus $L^{\wedge}_{ét,\mathrm{mot}}E \simeq L_{\mathbb{A}^1}E$. Since $\mathrm{Sp}_{\geq -1}$ is closed under colimits, it thus suffices to show that for $X \in \mathcal{S}\mathrm{m}_{\mathbb{F}_p}$ affine we have $(E/p)(X) \in \mathrm{Sp}_{\geq -1}$. This follows from [Bac20, Lemma 2.7(2)], using that affine $\mathbb{F}_p$-schemes have p-étale cohomological dimension ≤ 1 [GAV72, Théorème X.5.1]. □

Proof of Theorem A.1. Only the first statement requires proof. Since $\mathrm{cd}(\mathbb{F}_p) < \infty$, $\mathcal{SH}^{\wedge S^1}_{ét}(\mathbb{F}_p)$ is compactly generated by representables [Bac20, Corollary 5.7] and thus $L^{\wedge}_{ét,\mathrm{mot}} : \mathrm{Sp}(\mathcal{S}\mathrm{m}^{\wedge}_{S,ét}) \to \mathrm{Sp}(\mathcal{S}\mathrm{m}^{\wedge}_{S,ét})$ preserves colimits. Let $H\mathbb{Z} \in \mathrm{Sp}(\mathcal{S}\mathrm{m}^{\wedge}_{S,ét})_{\geq 0}$ denote the Eilenberg–MacLane spectrum. We seek to prove that $L^{\wedge}_{ét,\mathrm{mot}}(\mathbb{1}/p) = 0$. By Lemma A.2 we have $L^{\wedge}_{ét,\mathrm{mot}}(\mathbb{1}/p) \in \mathrm{Sp}(\mathcal{S}\mathrm{m}^{\wedge}_{S,ét})_{\geq -1}$, and hence $L^{\wedge}_{ét,\mathrm{mot}}(\mathbb{1}/p) = 0$ if and only if $L^{\wedge}_{ét,\mathrm{mot}}(\mathbb{1}/p) \wedge H\mathbb{Z} = 0$. Since $L^{\wedge}_{ét,\mathrm{mot}}$ preserves colimits and $H\mathbb{Z}$ lies in the subcategory generated under colimits by $\mathbb{1}$, we have $L^{\wedge}_{ét,\mathrm{mot}}(\mathbb{1}/p) \wedge H\mathbb{Z} \simeq L^{\wedge}_{ét,\mathrm{mot}}(H\mathbb{Z}/p)$. The forgetful functor $U : \mathrm{Mod}_{H\mathbb{Z}}(\mathrm{Sp}(\mathcal{S}\mathrm{m}^{\wedge}_{S,ét})) \to \mathrm{Sp}(\mathcal{S}\mathrm{m}^{\wedge}_{S,ét})$ commutes with $L^{\wedge}_{ét,\mathrm{mot}}$ (in fact $L^{\wedge}_{ét,\mathrm{mot}}$ is given by $L_{\mathbb{A}^1}$ in both categories, see the proof of Lemma A.2, and U preserves colimits [Lur17a, Corollary 4.2.3.5]), and the motivic localization of $\mathrm{Mod}_{H\mathbb{Z}}(\mathrm{Sp}(\mathcal{S}\mathrm{m}^{\wedge}_{S,ét}))$ is $DA^{\wedge S^1}_{ét}(\mathbb{F}_p, \mathbb{Z})$ (use [Lur18, Theorem 2.1.2.2]). Consequently we have reduced to proving that $DA^{\wedge S^1}_{ét}(\mathbb{F}_p, \mathbb{Z}/p) = *$, or equivalently that the unit of this symmetric monoidal category vanishes. This works using the standard argument, i.e. the fiber sequence $\mathbb{Z}/p \to \mathbb{G}_a \to \mathbb{G}_a$. □

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Mathematischen Institut, LMU Munich, Germany
Email address: tom.bachmann@zoho.com