Classification of surfaces

In these notes we work with irreducible projective varieties over $k = \mathbf{C}$.

1 Birational geometry

Definition 1. A rational map $f\colon X \dashrightarrow Y$ is a morphism $f\colon U \to Y$ for some dense open subset $U \subset X$. Two such maps are considered equal if they agree on a dense open subset of X. The *image* of f is the closure of $f(U)$. If the image of f is Y, then f is called *dominant*. The *function field* $K(X)$ of X is the set of rational maps $X \dashrightarrow \mathbf{A}^1$.

Clearly for any rational map there is a largest open subset on which it is defined. Note that it doesn't make sense to compose two rational maps in general. Thus, to form a category of rational maps, only dominant ones are considered. An isomorphism in this category is a *birational map*. This is easy:

Proposition 2. *The assignment $X \mapsto K(X)$ is an anti-equivalence between the category of (quasi-projective, irreducible) varieties with rational maps and that of finitely generated fields over k.*

However, there's more to birational geometry than field theory, since we are also interested in understanding how varieties can be birational to one another.

The general strategy for classifying varieties up to birational equivalence is to somehow single out varieties of a particular type within each birational class (called models), and then to classify all models. Depending on what is meant by "particular type", models might not exist or not be unique, and even if one is only interested in smooth varieties, models are usually not smooth. Typically, a strong condition on models makes their classification easier, but their existence harder. In all dimensions the following two types of models are considered:

- *Minimal models* are projective varieties with nef canonical bundle and only "terminal singularities".
- *Canonical models* are projective varieties with ample canonical bundle and only "canonical singularities".

Here I won't be talking about those singularities at all. It suffices to say that for surfaces, there are no terminal singularities, so that minimal models are smooth, and canonical singularities are easy to list (they are the so-called Du Val singularities). I recall the notions of nef and ample line bundles on smooth varieties in the next section. Because ample implies nef, one can expect minimal models to be easier to find than canonical ones, as we'll indeed observe for surfaces.

Example 3. For curves, one can successfully solve the birational classification by choosing models to be smooth projective curves: any curve is birational to such a curve, and any birational map between such curves is everywhere defined. For the above two choices of models, however, we have the following: minimal models are the smooth and projective curves of genus ≥ 1, while canonical models are the smooth and projective curves of genus ≥ 2. Thus, these models are unique but do not always exist.

The following is crucial:

Proposition 4. *Let $f\colon X \dashrightarrow Y$ be a rational map with X smooth and Y projective. Then there is a well-defined pullback map $f^*\colon \operatorname{Pic} Y \to \operatorname{Pic} X$ and for every $\mathcal{L} \in \operatorname{Pic} Y$ there is a map $f^*\colon H^0(Y, \mathcal{L}) \to H^0(X, f^*\mathcal{L})$. These have the expected functorial properties for composable rational maps.*

This is proved in two steps. First, one shows that the domain of definition U of f is such that $\operatorname{codim}(X - U) \geq 2$. Then one proves that any line bundle on such an open subset extends uniquely to a line bundle on X (for example using that on smooth varieties line bundles are represented by divisors). So one can first pull back $\mathcal{L}$ to U and then extend it uniquely to all of X. For the second statement, any rational section of $f^*\mathcal{L}$ defined on U extends uniquely to a regular section, since poles of rational functions have codimension 1.

2 Complements on line bundles

From now on, our varieties will be smooth (in addition to irreducible and projective). For the next definition, recall that intersection theory constructs a well-defined intersection pairing $(Y, Z) \mapsto Y.Z$ on the free abelian group generated by rational equivalence classes of irreducible closed subsets of X.

Definition 5. A divisor D on X is *nef* (resp. *ample*) if for every irreducible closed subset $Y \subset X$ of dimension r, $\deg(D^r.Y) \geq 0$ (resp. > 0).

Each notion defines a subset of the the Néron–Severi group $N^1(X)$ of divisors up to numerical equivalence which is closed under addition. A well-known result is that these subsets determine one another: the ample cone in $N^1(X) \otimes \mathbf{R}$ (i.e. the set of nontrivial linear combinations of ample classes with ≥ 0 coefficients) is the interior of the nef cone, which is in turn the closure of the ample cone. Another result due to Kleiman is that in the definition of nef it suffices to consider 1-dimensional Y's (which is the usual definition), but there doesn't seem to be a similar simplification for ample divisors, even on surfaces. Because numerical equivalence is coarser than rational equivalence, it makes sense to talk about nef and ample line bundles.

Let $\mathcal{L}$ be a line bundle on X admitting a nonzero global section, that is, $H^0(X, \mathcal{L}) \neq 0$. Then we get a rational map

$$\phi_{\mathcal{L}} : X \dashrightarrow \mathbf{P}(H^0(X, \mathcal{L}))$$

as follows. Choose a basis $s_0, \ldots, s_n$ of $H^0(X, \mathcal{L})$, and given $x \in X$ identify $\mathcal{L}_x$ with k. Then set

$$\phi_{\mathcal{L}}(x) = (s_0(x)s_0 : \ldots : s_n(x)s_n).$$

In fact any rational map $f : X \dashrightarrow \mathbf{P}^n$ to a projective space is obtained in this way, by choosing $\mathcal{L} = f^*(\mathcal{O}(1))$.

Definition 6. A line bundle $\mathcal{L}$ is *very ample* if $\phi_{\mathcal{L}}$ is an immersion.

A common definition of ampleness is:

Proposition 7. *A line bundle $\mathcal{L}$ is ample iff $\mathcal{L}^{\otimes n}$ is very ample for some $n \geq 1$.*

Ampleness is not a birational invariant. For example, if $\pi : \tilde{X} \to X$ is the blow-up of $\mathbf{P}^2$ at a point, then $-K_X$ is ample but $\pi^*(-K_X)$ is not: it is $E - K_{\tilde{X}}$ which has zero intersection with the exceptional divisor E.

In the case of an ample line bundle, one can thus recover X up to isomorphism as the image of $\phi_{\mathcal{L}^{\otimes n}}$ for any sufficiently divisble n (namely any multiple of an n for which $\mathcal{L}^{\otimes n}$ is very ample). If we want a birational version of ampleness we would only expect to recover the birational class of X. For arbitrary $\mathcal{L}$, one can in any case try to approximate X in this way. The rational maps $\phi_{\mathcal{L}^{\otimes n}}$ form an inverse system, with $\phi_{\mathcal{L}^{\otimes n}}$ factoring through $\phi_{\mathcal{L}^{\otimes m}}$ whenever $m = nd$. Consider the graded ring $R(\mathcal{L}) = \bigoplus_{n \geq 0} H^0(X, \mathcal{L}^{\otimes n})$. If $\mathcal{L}^{\otimes n}$ has sections for some $n \geq 1$, then by the general theory of Proj there is a dominant rational map $X \dashrightarrow \operatorname{Proj} R(\mathcal{L})$ through which each $\phi_{\mathcal{L}^{\otimes n}}$ factors. Roughly speaking, $\operatorname{Proj} R(\mathcal{L})$ is the limit of the images $\phi_{\mathcal{L}^{\otimes n}}(X)$ as n gets more and more divisible. For example, in the case where $R(\mathcal{L})$ is finitely presented, then $\operatorname{Proj} R(\mathcal{L})$ is the image of $\phi_{\mathcal{L}^{\otimes n}}$ as soon as n is a multiple of all degrees containing generators or relations. Ampleness means that $X \dashrightarrow \operatorname{Proj} R(\mathcal{L})$ is an isomorphism, which motivates the following definition.

Definition 8. A line bundle $\mathcal{L}$ on X is called *big* if $X \dashrightarrow \operatorname{Proj} R(\mathcal{L})$ is birational.

In other words, $\mathcal{L}$ is big if one can recover X up to birational equivalence as the image of $\phi_{\mathcal{L}^{\otimes n}}$ for sufficiently divisible n. Bigness is thus a substitute for ampleness which is a birational invariant: any ample line bundle is big, and if $f : X \dashrightarrow Y$ is a birational map, it induces an isomorphism $R(\mathcal{L}) \cong R(f^*(\mathcal{L}))$ by Proposition 4.

Proposition 9. *$\mathcal{L}$ is big iff* $\dim X = \dim \operatorname{Proj} R(\mathcal{L})$.

Proposition 10. *For any line bundle $\mathcal{L}$,* $\dim H^0(X, \mathcal{L}^{\otimes n}) = O(n^{\dim \operatorname{Proj} R(\mathcal{L})})$ *as $n \to \infty$.*

3 Kodaira dimension

If $f\colon X \to Y$ is a morphism between varieties of same dimension, top forms on Y pull back to top forms on X. In fact for any $n \geq 0$ we have a bundle map $f^*(\omega_Y^{\otimes n}) \to \omega_X^{\otimes n}$, whence $f^*\colon H^0(Y, \omega_Y^{\otimes n}) \to H^0(X, \omega_X^{\otimes n})$. By Proposition 4, this map exists even if f is just a rational map. Thus:

Key Fact 11. *$H^0(X, \omega_X^{\otimes n})$ and hence $R(\omega_X)$ are birational invariants of X.*

Assuming that $\operatorname{Proj} R(\omega_X)$ is nonempty, we get a dominant rational map

$$X \dashrightarrow \operatorname{Proj} R(\omega_X)$$

which is a birational equivalence if ω_X is big and an isomorphism if it is ample. This map is called the *Iitaka fibration.*

Definition 12. The *Kodaira dimension* $\kappa(X)$ of X is defined by

$$\kappa(X) = \dim \operatorname{Proj} R(\omega_X)$$

where by convention $\dim \varnothing = -\infty$.

Clearly, $\kappa(X) \leq \dim X$ with equality iff ω_X is big (in which case we say that X is *of general type*). One can rephrase the definition as follows: $\kappa(X)$ is the maximal dimension of the image of $\phi_{\omega_X^{\otimes n}}$ for $n \geq 1$.

Example 13. For a smooth projective curve C of genus g, the situation is as follows.

- If $g = 0$, then $-K_C$ is ample. In particular, $\omega_C^{\otimes n}$ has no sections for $n \geq 1$, so $\operatorname{Proj} R(\omega_C) = \varnothing$ and $\kappa(C) = -\infty$.
- If $g = 1$, $\omega_C \cong \mathcal{O}_C$ so $R(\omega_C) \cong k[x]$ and $\kappa(C) = 0$.
- If $g \geq 2$, then K_C is ample, so C is of general type.

In this example curves of general types are indeed the most common curves. We also see a relation between $\kappa(C)$ and the curvature of C. This is due rather to the behavior of K_C (if X is a Kähler manifold, $c_1(-K_X) \in H^2(X, \mathbf{R})$ is the class of the Ricci curvature of X). Still, in general, there are analogies between the Kodaira dimension and the curvature, such as the abundance of varieties of general type or of negative curvature.

The $-\infty$ convention is for the following result, which follows from Proposition 10.

Proposition 14. *$\kappa(X \times Y) = \kappa(X) + \kappa(Y)$.*

4 Surfaces

The problem of understanding how to move within a given birational class has a very neat answer for surfaces (as always, smooth and projective):

Theorem 15. *If $f\colon X \dashrightarrow Y$ is a birational equivalence between surfaces, there exists a variety Z and morphisms $Z \to X$ and $Z \to Y$ obtained by successively blowing up points. Moreover, if f is everywhere defined, we can choose $Z \to X$ to be the identity.*

Example 16. Consider the obvious birational map $\mathbf{P}^1 \times \mathbf{P}^1 \dashrightarrow \mathbf{P}^2$. Then a variety Z as in the theorem is given by the 27-line cubic (a cubic embedded in $\mathbf{P}^3$): Z can be obtained from $\mathbf{P}^2$ be blowing up 6 points in general position, but it can also be obtained from $\mathbf{P}^1 \times \mathbf{P}^1$ by blowing up 5 points.

The following is Castelnuovo's criterion:

Theorem 17. *Let C be a curve in a surface X. The following are equivalent.*

1. *There exists a smooth surface X' such that C is the exceptional divisor of a blowup at a point $X \to X'$.*

2. *$C \cong \mathbf{P}^1$ and $C^2 = -1$ (C is an* exceptional curve*).*

Definition 18. A surface X is called *relatively minimal* if any everywhere defined birational map $X \to Y$ is an isomorphism.

It is obvious from the previous theorems that a surface is relatively minimal iff it does not contain an exceptional curve. Relatively minimal surfaces are often just called minimal, but this only agrees with the definition of minimal model given in the first section in nonnegative Kodaira dimension (Theorem 20).

Thus, starting from a surface X, one can successively contract exceptional curves. In fact this process has to stop:

Proposition 19. *Let X be a surface. Given any chain $X = X_0 \to X_1 \to \ldots \to X_n$ in which each $X_{i-1} \to X_i$ is a blowup of X_i at a point, $n \leq \dim H^1(X, \Omega^1_X)$.*

One proof goes as follows. Denote by E_i the exceptional divisor of $X_{i-1} \to X_i$ pulled back to X. By the rules giving the intersection product in a blowup, we have $\deg(E_i.E_j) = -\delta_{ij}$. Now we use the group homomorphism

$$d\log\colon \operatorname{Pic} X \cong H^1(X, \mathcal{O}^*_X) \to H^1(X, \Omega^1_X)$$

which is known to transform the intersection pairing into the Serre duality pairing. The theorem follows because the latter is nondegenerate.

This result implies that any surface has a relatively minimal model. Furthermore:

Theorem 20. *Let X' be a relatively mimimal model of a surface X. Then $K_{X'}$ is nef iff $\kappa(X) \geq 0$. In this case, for any relatively minimal model X'' of X, the composition of birational maps $X' \dashrightarrow X \dashrightarrow X''$ is an isomorphism.*

When $\kappa(X) = -\infty$, there may be several relatively minimal models. For example, $\mathbf{P}^2$ and $\mathbf{P}^1 \times \mathbf{P}^1$ are both relatively minimal. Because a surface X with an exceptional curve C cannot have a nef canonical bundle ($\deg(K_X.C) = -1$ by the adjunction formula), the minimal models as defined in the first section are precisely the relatively minimal surfaces with nonnegative Kodaira dimension. In particular, minimal models are unique. For canonical models, one has the following:

Theorem 21. *If X is a surface of general type, then* $\operatorname{Proj} R(\omega_X)$ *is the unique canonical model of X.*

I now give a summary of the classification of surfaces due to Enriques.

- Case $\kappa = -\infty$. This class contains *rational* and *ruled* surfaces. A rational surface is one birational to $\mathbf{P}^2$. A surface is ruled if it is the projective bundle of lines in some 2-dimensional vector bundle over a smooth curve C (birational to $C \times \mathbf{P}^1$ by local triviality). A system of representative of the birational classes is given by the surfaces $C \times \mathbf{P}^1$ where C ranges over smooth curves. The relatively minimal rational surfaces are $\mathbf{P}^2$, $\mathbf{P}^1 \times \mathbf{P}^1$, and the ruled surfaces $\mathbf{P}(\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(n))$ for $n \geq 2$. All nonrational ruled surfaces are relatively minimal.

- Case $\kappa = 0$. This class contains:

  - *Abelian surfaces.* These are surfaces with a group structure, and are exactly those quotients of $\mathbf{C}^2$ by a lattice which are algebraic. The canonical bundle is trivial, so these surfaces are all minimal by the adjunction formula.

  - *K3 surfaces.* Surfaces with trivial canonical bundle that are not abelian. An example is a degree 4 surface in $\mathbf{P}^3$. They are all minimal.

  - *Enriques surfaces.* They are quotients of K3 surfaces by a free action of $\mathbf{Z}/2\mathbf{Z}$. Their canonical bundle is not trivial but $\omega_X^{\otimes 2}$ is. Any such surface X arises as an elliptic bundle over $\mathbf{P}^1$, that is, there is a surjective map $X \to \mathbf{P}^1$ whose fibers are generically elliptic curves.
  - *Hyperelliptic surfaces.* These are nontrivial elliptic bundles over elliptic curves. They are quotients of abelian surfaces by finite group actions.

- Case $\kappa = 1$. All other elliptic bundles over curves (in this case the bundle map can be taken to be the Iitaka fibration).

- Case $\kappa = 2$. Surfaces of general type. These are very numerous and hard to classify. For instance, any bundle over a curve of general type (genus ≥ 2) whose fibers are of general type is in this class, as is any hypersurface of degree $d \geq n+2$ in $\mathbf{P}^n$ (in this case $\omega_X = \mathcal{O}_X(d-n-1)$ is even ample, so it is a canonical model).