SEQUENTIAL VS. SYMMETRIC SPECTRA

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Let $\mathcal{C}$ be a presentably symmetric monoidal ∞-category and let $X \in \mathcal{C}$. Following [AI23, Section 1], we write $\mathrm{Tel}_X(\mathcal{C})$ for the ∞-category of sequential X-spectra and $\mathrm{Sp}_X(\mathcal{C})$ for the ∞-category of symmetric X-spectra. The former is a $\mathcal{C}$-module while the latter is a commutative $\mathcal{C}$-algebra in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$.

There is a conservative forgetful functor $\mathrm{Sp}_X(\mathcal{C}) \to \mathrm{Tel}_X(\mathcal{C})$ in $\mathcal{P}\mathrm{r}^{\mathrm{R}}$, which is well known to be an isomorphism when the cyclic permutation of $X^{\otimes n}$ is the identity for some $n \geq 2$, but not in general. The following rather surprising result, which we learned from Jacob Lurie, shows that when $\mathcal{C}$ is stable, being symmetric is merely a property of a sequential spectrum:

Proposition 1 (Lurie)**.** *Let $\mathcal{C}$ be a stable presentably symmetric monoidal ∞-category and let $X \in \mathcal{C}$. Then the forgetful functor* $\mathrm{Sp}_X(\mathcal{C}) \to \mathrm{Tel}_X(\mathcal{C})$ *is fully faithful. Its essential image is* $\mathrm{Tel}_X(\mathcal{C}')$*, where $\mathcal{C}'$ is the full subcategory of $\mathcal{C}$ consisting of all objects Y such that the action of a 3-cycle on the internal mapping object* $\mathrm{Hom}(X^{\otimes 5}, Y)$ *is homotopic to the identity.*

Proof. Let A be a pointed connected anima such that $\pi_1(A)$ is a perfect group and let $a\colon A \to A^+$ be Quillen's plus construction. For a local system $L\colon A \to \mathcal{C}$, the following are equivalent:

(i) L factors through the epimorphism a;
(ii) L induces the trivial map on π_1;
(iii) $L(*) \to \mathrm{colim}(L|\,\mathrm{fib}(a))$ is an isomorphism;
(iv) the unit map $L \to a^*a_!L$ is an isomorphism;
(v) $\lim(L|\,\mathrm{fib}(a)) \to L(*)$ is an isomorphism;
(vi) the counit map $a^*a_*L \to L$ is an isomorphism.

Since $\pi_1(A)$ is perfect, the equivalence (i) $\Leftrightarrow$ (ii) is the universal property of the plus construction. The implications (iii) $\Leftrightarrow$ (iv), (v) $\Leftrightarrow$ (vi), (iv) $\Rightarrow$ (i), and (vi) $\Rightarrow$ (i) are obvious. Finally, if L factors through A^+, then the maps $L(*) \to \mathrm{colim}(L|\,\mathrm{fib}(a)) \simeq \mathrm{fib}(a) \otimes L(*)$ and $L(*)^{\mathrm{fib}(a)} \simeq \lim(L|\,\mathrm{fib}(a)) \to L(*)$ are isomorphisms since $\mathrm{fib}(a)$ is acyclic and $\mathcal{C}$ is stable.

We now apply this with A the delooping of the alternating group A_5, which is perfect, and L the action of A_5 on $X^{\otimes 5}$. Since A_5 is generated by 3-cycles, the equivalence (ii) $\Leftrightarrow$ (v) applied to $\mathrm{Hom}(-, Y) \circ L$ shows that $\mathcal{C}'$ is exactly the symmetric monoidal left Bousfield localization of $\mathcal{C}$ at the map $L(*) \to \mathrm{colim}(L|\,\mathrm{fib}(a))$. This map is an isomorphism if X is invertible, by the implication (ii) $\Rightarrow$ (iii). Hence, the functor $\mathrm{Sp}_X(\mathcal{C}) \to \mathrm{Sp}_X(\mathcal{C}')$ in $\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$ is an isomorphism by comparison of universal properties. By (iii) $\Rightarrow$ (ii), the cyclic permutation on $X^{\otimes 5}$ becomes the identity in $\mathcal{C}'$, so that the functor $\mathrm{Tel}_X(\mathcal{C}') \to \mathrm{Sp}_X(\mathcal{C}')$ in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ is an isomorphism. The following commutative square in $\mathcal{P}\mathrm{r}^{\mathrm{R}}$ completes the proof:

$$\begin{array}{ccc} \mathrm{Sp}_X(\mathcal{C}') & \xrightarrow{\sim} & \mathrm{Tel}_X(\mathcal{C}') \\ \wr\downarrow & & \downarrow \\ \mathrm{Sp}_X(\mathcal{C}) & \longrightarrow & \mathrm{Tel}_X(\mathcal{C}). \end{array}$$

□

References

[AI23] T. Annala and R. Iwasa, *Motivic spectra and universality of K-theory*, 2023, arXiv:2204.03434

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